Euclidean Constructions
The Straightedge and the Compass

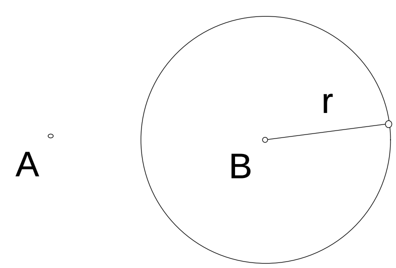
While there is only one type of straightedge — a ruler without markings — there are two types of compasses. The Greeks, from Plato, use dividers, or what we now call a collapsing compass. This compass loses the radius once the instrument is moved. This would mean that you could not construct a circle and then move the compass and construct a circle of the same radius. The constructions are all related to Euclid's first three axioms:

1. *To draw a straight line from any point to any point*.
2. *To produce a finite straight line continuously in a straight line*.
3. *To draw a circle with any center and radius*.

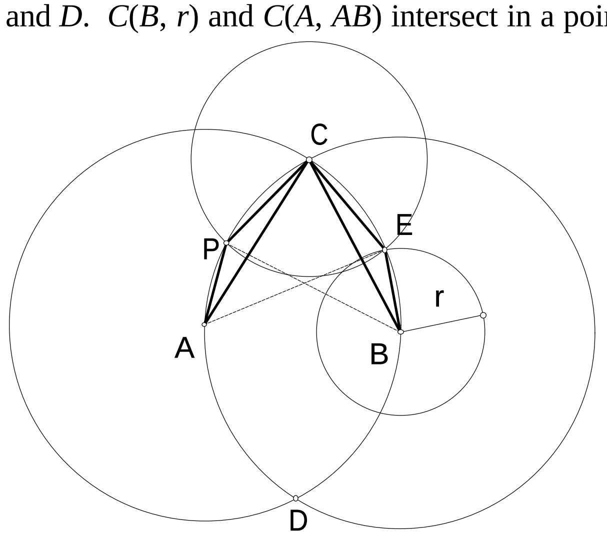
The straightedge is thought of as having no length markings because none of these postulates provides us with the ability to measure lengths. Thus you may use the straightedge to connect a pair of given points or extend a given line segment arbitrarily in a straight line. However, you cannot use the straightedge to extend a line segment 3 inches.

Since the straightedge is the same, this does not bother us. However, since the Euclidean compass is different from the modern, or fixed, compass, we do have some concerns. Can we make the same constructions? Are the rules the same for both compasses? Well, we can.

Theorem: The Compass Equivalence Theorem. *A circle C(B,r) can be congruently copied (using only a straightedge and collapsing compass) so that a given point A serves as the center of the copy*.



Consider the given circle and a point *A* as shown to the right. We need to construct a circle of radius *r* centered at *A*, using only a straightedge and collapsing compass. First construct the circle centered at *A* with radius *AB* and the circle centered at *B* with radius *BA*. These circles intersect in two points. Call the points *C* and *D*. *C*(*B*, *r*) and *C*(*A*, *AB*) intersect in a point *E*. Consider the circle *C*(*C*, *CE*). It will intersect *C*(*B*,*BA*) in a point *P*. We claim that $AP \cong r$.



$\Delta PCB \cong \Delta ECA$ by SSS. Thus, $\mathbf{R}PCB \cong \mathbf{R}ECA$. Thus,

$\mathbf{R}PCB - \mathbf{R}ACB = \mathbf{R}ECA - \mathbf{R}ACB$. Thus

$\mathbf{R}PCA \cong \mathbf{R}ECB$.

$\cong$ and AC BC

$\Delta APC \cong \Delta BEC$, which implies that $AP \cong BE \cong r$.

Construction 1: Construct the perpendicular bisector of a line segment.

Construction 2: Construct a perpendicular line from a given point to a line not containing that point.

Construction 3: Construct a line perpendicular to a given line at a given point **P** on *l*.

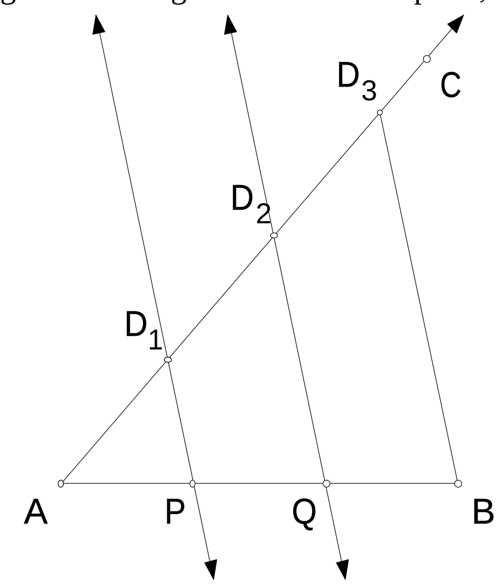
Construction 4: Construct an angle bisector.

Construction 5: Copy a given angle to a given ray with a given vertex.

Construction 6: Given a line *l* and a point *P* not on *l*, construct a line through *P* parallel to the given line.

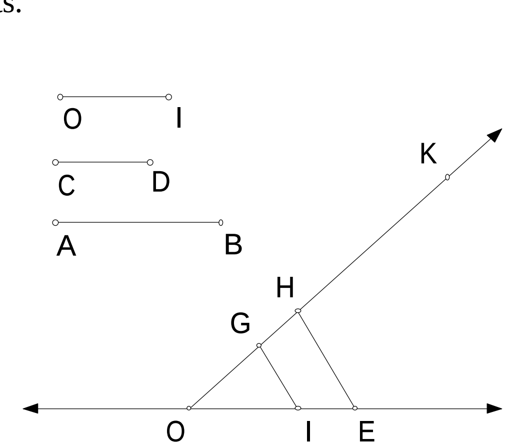
Construction 7: Partition a line segment into three (or more) congruent pieces.

While the previous constructions are quite familiar, this one may not be quite so well described in your high school geometry course. It is quite simple and we will illustrate it with $n = 3$. Let *AB* denote the segment that we wish to divide into three congruent sub-segments. Choose a point, *C*, not on the line $\overleftrightarrow{AB}$. We would prefer that you choose *C* so that the angle $\angle CAB$ is an acute or right angle, but this is not necessary. Choose a point, D_1, on the ray $\overrightarrow{AC}$. This could be the point C, but need not be. Normally, we would choose the segment, AD_1, to have unit length, but again this is not necessary. Copy AD_1 twice more on the ray $\overrightarrow{AC}$, to get two more segments, D_1D_2 and D_2D_3, each of which is congruent to AD_1. Thus the final segment has the property that $AD_3 \cong 3AD_1$. Now, construct the segment D_3B. Construct the lines through D_2 and D_1 parallel to the line $\overleftrightarrow{D_3B}$. Each will intersect AB in a point (by Pasch's Theorem). Call these points *P* and *Q*. Then, $AP \cong PQ \cong QB$ and you have trisected the segment.



Construction 8: Represent the product of two segments.

This construction is very much like the previous construction. We will have to use similar triangles. Suppose that *AB* has length *x* and *CD* has length *y*. How do we construct a segment that has length *xy*? We will need a unit length. Choose a number line $\overleftrightarrow{PQ}$ scaled with a unit length *OI*, starting at the origin *O*. Copy *AB* onto *OE*. Now choose a point *K* not on $\overleftrightarrow{PQ}$ and construct $\overrightarrow{OK}$. Copy *CD* onto $\overrightarrow{OK}$ originating at *O* and terminating at the point *G*. Construct *GI*. Construct a line through *E* parallel to *GI*. This line inter-



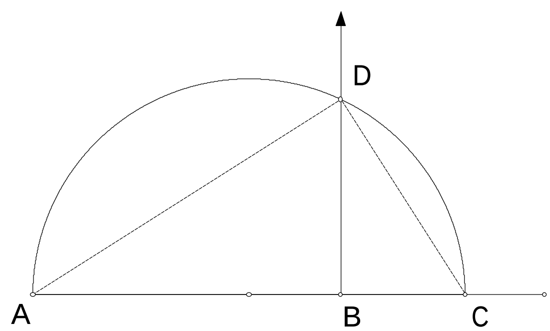
sects $\overrightarrow{OK}$ at a point H. Then, the length of OH is xy. This is true because in the similar triangles ΔGOI and ΔHOE we have:

$$\frac{GO}{HO} = \frac{OI}{OE}$$
$$\frac{y}{HO} = \frac{1}{x}$$
$$HO = xy$$

Construction 9: Represent the quotient of two segments.

This uses the same construction technique but changes the roles of the segments. Can you determine which segments will be which?

What other numbers can be represented by segments, beginning with a unit segment and using only the straightedge and compass to construct other segments? This is the set of *constructible numbers*. How big is it? Clearly, we can perform addition, subtraction, multiplication and division. This means that we can construct all integers and rational numbers. Can we construct more? We suspect so. We can surely construct $\sqrt{2}, \sqrt{3}$ and others. Given a number *a*, that is constructible, can we construct $\sqrt{a}$. First, assume that AB has length a and BC has length 1. We construct a circle with AC as a diameter and let the line perpendicular to AC passing through B intersect the circle at the point D. Then, $\Delta ADB \sim \Delta DCB$ and



$$\frac{AB}{BD} = \frac{DB}{BC}$$
$$\frac{a}{BD} = \frac{BD}{1}$$
$$a = (BD)^2$$
$$BD = \sqrt{a}$$

The theory of constructible numbers is pretty well covered by the next two theorems.

Theorem: *The use of a straightedge alone can never yield segments of numbers outside the original number field.*

If a is an element of the number field so that $\sqrt{a}$ is not an element of our number field, then our construction technique can extend this number field by making $\sqrt{a}$. Then we can construct all numbers of the form $b + c\sqrt{a}$, where *b* and *c* are in the number field. Now, the set of all numbers of this form also forms a field, called an extension field of the original field of numbers.

Theorem: *A real number is constructible if and only if it is of the form $b + c\sqrt{a}$, where each of a, b, and c are either rational numbers or are constructed from repeated extraction of square roots of rational numbers.*

The best way to state this is that the number lies in an extension field of the rational numbers of degree 2^n.

Construction 10: Construct a square with a given side.

Construction 11: Construct an equilateral triangle with a given side.

Construction 12: Construct a regular pentagon with a given side.

Construction 13: Construct a regular *n*-gon with a given side. Is this possible for all *n*? For which *n* is it possible?

Gauss proved the following result:

Theorem: ***(Gauss)*** *A regular polygon can be inscribed in a circle by means of a straightedge and compass alone if and only if the number of sides, n, can be expressed as* $n = 2^k \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_m$, *for a nonnegative integer k and each* p_i *a distinct prime of the form* $2^{2^r} + 1$, *for* $r \geq 0$.

Some of the regular polygons that are constructible, according to this theorem, are those with 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, or 24 sides. Note that the theorem does not tell us how to do the construction, only that it can or cannot be done.